International Journal of Theoretical Physics, Vol. 44, No. 12, December 2005 (© 2005)
DOI: 10.1007/s10773-005-8023-1

The Canonical Topology on a Meet-Semilattice

Isar Stubbe[1]

Received ; accepted

Considering the lattice of properties of a physical system, it has been argued elsewhere that—to build a calculus of propositions having a well-behaved notion of disjunction (and implication)—one should consider a very particular frame completion of this lattice. We show that the pertinent frame completion is obtained as sheafification of the presheaves on the given meet-semilattice with respect to its canonical Grothendieck topology, an explicit description of which is easily given. Our conclusion is that there is an intrinsic categorical quality to the notion of "disjunction" in the context of property lattices of physical systems.

KEY WORDS: meet-semilattice; topology; sheaf; frame completion.

PACS: 02.10.Ab; 03.65.Fd.

1. DISTRIBUTIVE JOINS IN A PROPERTY LATTICE ARE DISJUNCTIONS

Following Piron (1972, 1976, 1990) in his operational approach to (quantum) physics, a physical system is described by its properties, the actuality of each property being tested by a definite experimental project. The collection of properties (actual or not) of a system forms a complete lattice $(L, \leq)$: the order relation in L is the "implication of actuality" of properties ($a \leq b$ in L means that b is actual whenever a is), and it is a matter of fact that the infimum in L is the conjunction of properties ($\wedge_i a_i$ in L is actual if and only if every a_i is actual). The state of a system is defined as the collection of all of its actual properties; but it is easily seen that one can identify a state $\varepsilon \subset L$ with $p_\varepsilon := \wedge\varepsilon \in L$. When denoting by S the set of all possible states of a given physical system, Aerts (1982) put forward that S is a *space* rather than just a set, its structure coming from the so-called "Cartan map"

$$\mu : L \to \mathcal{P}(S) : a \mapsto \{\varepsilon \in S \mid p_\varepsilon \leq a\}.$$

[1] Département de Mathématique, Université de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium; e-mail: i.stubbe@math.ucl.ac.be.

The supremum in the property lattice $(L, \leq)$ is given by the "infimum of upper bounds," so it is a mathematical rather than an operational ingredient. In general it is quite different from a disjunction: $a \vee b$ in L can be actual without either a or b being actual! However, in certain situations one would like to have a formal equivalent of "*a or b*," as for instance in the phrase: were we to measure a spin–$\frac{1}{2}$ particle, then after the measurement we would find a particle with "spin up *or* spin down." Of course, "spin up *or* spin down" is not a property of the physical system (because it cannot be tested!); it is rather a *proposition* about the properties. It was proposed by Coecke (2001) to devise a new structure containing not only all properties of the system but also all disjunctions of the properties. Because of the operational meaning of the infimum of properties (it is their conjunction), the embedding of $(L, \leq)$ in this new structure should preserve all infima; and clearly, would $\vee_i a_i$ be a disjunction in L, then in the new structure the property $\vee_i a_i$ should be identified with the disjunction of the a_i. More precisely, under some mild conditions on the Cartan map $\mu : L \to \mathcal{P}(S)$ (satisfied in all known examples), it is shown in *loc. cit.* that, for a subset $A \subseteq L$, the supremum $\vee A$ in L is a disjunction (i.e. $\mu(\vee A) = \cup_{a \in A} \mu(a)$) if and only if for every $x \in L$ one has that $x \wedge (\vee A) = \vee(x \wedge A)$. That is to say, in a property lattice $(L, \leq)$ the distributive joins are exactly the disjunctions. Therefore, "adding to a property lattice those disjunctions that did not already exist, and keeping the conjunction" means "to embed the lattice $(L, \leq)$ in a frame[2] such that all meets and all distributive joins are preserved." This can indeed be done; in the next section we go through the elementary description of the required frame completion, in the somewhat more general case of a meet-semilattice $(P, \leq)$ rather than a complete lattice $(L, \leq)$. The implementation of these results in the operational approach to physics is discussed elsewhere (see, for instance, Coecke, 2001), so we do not bother to do so here.

Our thesis in this paper is that the frame completion described above can and should be seen as an instance of the far reaching concept of "sheafification of presheaves on a small category endowed with a Grothendieck topology." In our opinion, specializing general categorical principles to the case of posets is of more than merely aesthetic interest, for categorical constructions often give a better understanding of the main concepts, even in poset theory. But this development also provides more evidence in favor of the viewpoint that there is an intrinsic categorical quality to quantum logic and related fields in theoretical physics. In fact, several recent as well as older publications indicate that category theory is as much a tool for the theoretical physicist as for the working mathematician (see, for instance, Coecke *et al.*, 2001; Coecke and Stubbe, 1999a,b; Moore, 1995; Stubbe, 2001).

[2] A *frame* is a complete lattice in which finite meets distribute over arbitrary joins (see Johnstone, 1982).

2. ELEMENTARY DESCRIPTION OF THE PERTINENT UNIVERSAL FRAME COMPLETION

Throughout this text, the phrase "meet-semilattice" refers, as usual, to a commutative monoid $(P, \wedge, 1)$ in which each element is idempotent. Put differently, it is a partial order $(P, \leq)$ in which all finite (thus also empty) meets exist. A homomorphism of such objects is simply a homomorphism of monoids: it preserves all finite (and empty) meets; an obvious category MSLat results. Of course, every homomorphism is monotonic, but in general a monotonic mapping need not be a homomorphism. We will use the notation Monot for the category of monotonic maps between meet-semilattices.

Definition 1. Given a (possibly infinite) family $\{x_i\}_{i\in I}$ of elements in a meet-semilattice P, it is said to have a distributive join (and, for short, the family is said to be distributive) if

(a) its join $\vee_i x_i$ exists in P,
(b) for every $x \in P$ one has that $x \wedge (\vee_i x_i) = \vee_i (x \wedge x_i)$ (so in particular the join on the right-hand side must exist).

A mapping between two meet-semilattices, say $f : P \to Q$, is said to preserve distributive joins if, whenever a family $\{x_i\}_{i\in I}$ is distributive in P, then $\{f(x_i)\}_{i\in I}$ is distributive in Q and moreover $f(\vee_i x_i) = \vee_i f(x_i)$.

Clearly, such a morphism $f : P \to Q$ is always monotonic; it suffices to notice that $x \leq y$ in P implies that $\{x, y\}$ has a distributive join in P, such that $f(y) = f(\vee\{x, y\}) = \vee\{f(x), f(y)\}$ which implies that indeed $f(x) \leq f(y)$. Therefore, as we use the notation Monot for the category of monotonic maps between meet-semilattices, we will use $\mathsf{Monot}_{\mathsf{dis}}$ for the subcategory determined by morphisms that preserve distributive joins. In the same vein, $\mathsf{MSLat}_{\mathsf{dis}}$ will denote the subcategory of MSLat determined by those morphisms that preserve (finite meets and) distributive joins.

A meet-semilattice in which all joins exist and are distributive is by definition a frame. The category Frm of frames is the full subcategory of $\mathsf{MSLat}_{\mathsf{dis}}$ determined by those meet-semilattices that are frames. Clearly Frm is also a subcategory of MSLat—however not fully so.

The following result can be found in any standard reference on frame theory (see, for instance, Johnstone, 1982).

Proposition 1. Frm *is a monoreflective subcategory of* MSLat.

In other words, for every meet-semilattice P there exists a frame H_P (necessarily unique up to isomorphism) and an embedding $i_P : P \hookrightarrow H_P$ in the category MSLat (so, in particular, i_P is an injection that preserves finite meets) enjoying the universal property that for any other frame H and any other MSLat-morphism $g : P \to H$ there exists a unique Frm-morphism $g_{\mathsf{ext}} : H_P \to H$ that extends g along i_P: $g_{\mathsf{ext}} \circ i_P = g$.

A subset D of a poset P is said to be a downset if, whenever $x \leq y$ in P and $y \in D$, also $x \in D$. Clearly P can be embedded in $\mathsf{Dwn}(P)$—the latter denoting the poset of downsets of P, ordered by inclusion—simply by sending each $x \in P$ to the principal downset $\downarrow x = \{t \in P \mid t \leq x\}$:

$$\downarrow(-) : P \hookrightarrow \mathsf{Dwn}(P) : x \mapsto \downarrow x. \tag{1}$$

The poset $\mathsf{Dwn}(P)$ is a frame: meet is given simply by intersection and join by union, so it is obvious that finite meets distribute over arbitrary joins. And the inclusion $\downarrow(-) : P \hookrightarrow \mathsf{Dwn}(P)$ clearly preserves all meets that happen to exist in P.

In particular if P is a meet-semilattice then (1) is a morphism in MSLat, and it is easily seen to be the universal frame completion in MSLat, cf. Proposition 1.

But—alas!—the embedding in diagram (1) is not a morphism in $\mathsf{MSLat}_{\mathsf{dis}}$: should P have distributive joins, then these are not necessarily preserved by the downset inclusion! This defect is, however, easily repaired as shown by the following theorem that was first proved in a somewhat different setting in Bruns and Lakser (1970).

Proposition 2. Frm *is a full monoreflective subcategory of* $\mathsf{MSLat}_{\mathsf{dis}}$.

The only—but crucial!—difference with the previous theorem is the word *full*. As we already observed, Frm is indeed a *full* subcategory of $\mathsf{MSLat}_{\mathsf{dis}}$. Thus this theorem says that for every poset P with finite meets there exists a universal frame completion $i_P : P \hookrightarrow H_P$ in the category $\mathsf{MSLat}_{\mathsf{dis}}$—so, in particular, H_P is a frame and i_P is an injection that not only preserves finite meets but also those distributive joins that happen to exist in P.

For an elementary description of the required universal completion for Proposition 2, one can use the technique of congruences. Suppose H is a frame; an equivalence relation $R \subseteq H \times H$ is a congruence if $(x, y), (x', y') \in R$ implies $(x \wedge x', y \wedge y') \in R$ and $\{(x_k, y_k)\}_{k \in K} \subseteq R$ implies $(\vee_k x_k, \vee_k y_k) \in R$. The collection of congruences on a given frame H is a complete lattice, in particular, the smallest congruence containing a given relation $R \subseteq H \times H$ (i.e. the congruence "generated by R") exists.

Consider now, for a meet-semilattice P, the following binary relation on $\mathsf{Dwn}(P)$:

> whenever $\{x_k\}_{k \in K}$ is a family with distributive join in P,
> we declare that $\cup_k \downarrow x_k$ is in relation with $\downarrow(\vee_k x_k)$.

The quotient of $\mathsf{Dwn}(P)$ by the congruence generated by this relation will be denoted by $\mathsf{Dwn}_{\mathsf{dis}}(P)$; its elements can be identified by those downsets that are closed under distributive joins: whenever $\{x_i\}_{i\in I}$ is a family with distributive join in P whose elements x_i are in $D \in \mathsf{Dwn}_{\mathsf{dis}}(P)$, then also $\vee_i x_i$ is in D. The universal frame completion in $\mathsf{MSLat}_{\mathsf{dis}}$ is now the factorization of the embedding $P \hookrightarrow \mathsf{Dwn}(P) : x \to \downarrow x$ over this $\mathsf{Dwn}_{\mathsf{dis}}(P)$.

Remark that $\mathsf{Dwn}_{\mathsf{dis}}(P)$ being a quotient of $\mathsf{Dwn}(P)$ in Frm means in particular that there exist an embedding and a surjection as in

$$\mathsf{Dwn}_{\mathsf{dis}}(P) \underset{\hookrightarrow}{\twoheadleftarrow} \mathsf{Dwn}(P) \tag{2}$$

forming a Galois pair, the surjection being left adjoint to the injection, thus the former preserving suprema and the latter infima; moreover, the surjection preserves finite (and empty) meets.

3. INTRINSIC CATEGORICAL QUALITY OF DISJUNCTIONS

The rest of this paper is concerned with a categorical analysis of the key concepts behind the frame completion described above; it turns out that this is all about Grothendieck topologies on a (small) category and toposes of (pre)sheaves on the resulting site. Our references on category theory are of Borceux (1994) MacLane (1971) or Borceux and Stubbe (2000) for a concise introduction to the subject; for the more specific subject of toposes, the reader may consult (MacLane and Moerdijk 1992).

To establish our notions, we shall briefly recall some definitions and results from category theory. A (contravariant) presheaf on a category $\mathcal{C}$ is a functor $F: \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ from the opposite of the category $\mathcal{C}$ into the category Set of sets and functions. Taking such presheaves on $\mathcal{C}$ as objects and natural transformations as arrows, gives the topos $\mathsf{PreSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$. Any object X of the category $\mathcal{C}$ determines a representable presheaf $Y_X: \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$: it sends a morphism $f: A \to B$ in $\mathcal{C}$ to the function

$$- \circ f: \mathcal{C}(B, X) \to \mathcal{C}(A, X): g \mapsto g \circ f$$

in Set. (As usual, $\mathcal{C}(B, X)$ stands for the set of morphisms in $\mathcal{C}$ with domain B and codomain X; likewise for $\mathcal{C}(A, X)$.) The so-called Yoneda lemma asserts that for any category $\mathcal{C}$, any presheaf $F: \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ and any object $X \in \mathcal{C}$, there is a bijection between on the one hand the set of natural transformations from Y_X to F and on the other hand the set FX. For a small category $\mathcal{C}$, i.e. one that has a set of objects, this implies that $\mathcal{C}$ can be fully faithfully embedded in the topos of presheaves on $\mathcal{C}$:

$$Y: \mathcal{C} \hookrightarrow \mathsf{PreSh}(\mathcal{C}).$$

Moreover, this *Yoneda embedding* preserves all limits that happen to exist in $\mathcal{C}$, but not so for colimits: it does map cocones onto cocones, but in general "with loss of eventual universality." Still, one can try to describe a restriction of the codomain of the embedding, aiming at a preservation of (certain types of) colimits (that may or may not exist) in $\mathcal{C}$. Moreover, one might want such a restriction to inherit (some or all of the) nice properties of the ambient topos $\mathsf{PreSh}(\mathcal{C})$.

The theory of Grothendieck toposes provides an answer to this question: for every Grothendieck topology J on the category $\mathcal{C}$, the sheaves on the site $(\mathcal{C}, J)$ form a topos, and the inclusion functor $\mathsf{Sh}(\mathcal{C}, J) \hookrightarrow \mathsf{PreSh}(\mathcal{C})$ from the topos of sheaves on the site $(\mathcal{C}, J)$ to the topos of presheaves on $\mathcal{C}$ has a left adjoint that preserves finite limits. Such a topology J on a (small) category $\mathcal{C}$ is said to be subcanonical if all representable presheaves are in fact sheaves. So saying that J is subcanonical, is saying that the Yoneda embedding $\mathrm{Y} : \mathcal{C} \to \mathsf{PreSh}(\mathcal{C})$ corestricts to an embedding $\mathcal{C} \to \mathsf{Sh}(\mathcal{C}, J)$, i.e. $\mathcal{C}$ is a subcategory of the Grothendieck topos $\mathsf{Sh}(\mathcal{C}, J)$. As it can be proven that the collection of topologies on a category form a complete lattice, one can consider the finest topology for which all representable presheaves are sheaves, which is called the *canonical topology*. Intuitively it is clear that "the finer the topology J, the fewer the sheaves on the site $(\mathcal{C}, J)$," so that J being the canonical topology on $\mathcal{C}$ will produce the "smallest" topos $\mathsf{Sh}(\mathcal{C}, J)$ in which $\mathcal{C}$ can be embedded by (a corestriction of) the Yoneda functor. Whereas the existence of a canonical topology is evident, it is in general quite hard to write down an explicit description.

In this section, we adapt this essentially categorical insight to the situation at hand. Any poset $(P, \leq)$ can be thought of as a category: its objects are the elements of P, and there is an arrow $x \to y$ precisely when $x \leq y$. This means, in particular, that there is at most one arrow from x to y, and therefore such a category may be called "thin." To say that the poset has finite (and empty) meets, is to say that as a category it has finite limits (and a terminal object). Clearly a functor between two posets-seen-as-categories is just a monotonic map. Therefore, working with the category Monot and its derivatives such as $\mathsf{Monot}_{\mathsf{dis}}$ and Frm and so on, can be considered as "doing thin category theory." This idea of looking at the theory of ordered structures as a "thin category theory" can be made precise by means of enriched category theory. Indeed, when taking 2 to be the poset $\{0 \leq 1\}$ viewed as category, the theory of (small) 2-enriched categories is precisely the theory of preordered sets and monotonic mappings. Crucial to this point of view is that the hom-sets of a thin category (i.e. a poset-seen-as-category) are objects of 2: for any $x, y \in P$ there is either 0 or 1 arrow from x to y. For an ordinary category $\mathcal{C}$, the hom-sets are sets, that is, objects of the category Set. Therefore, when adapting (or restricting) ordinary category theory to the theory of (pre)ordered sets, one must not only replace "category" by "poset," and "functor" by "monotonic map," but also "Set" by "2"!

Our program is to explicitly describe the "topology of distributive covers" on a given poset P with finite meets, viewed as a thin category, and then to prove that it is the canonical topology in the sense explained above. Whereas the "thin presheaves" on P coincide with its downsets, the "thin sheaves with respect to the canonical topology" on P coincide with those downsets that are closed under distributive joins, such that eventually the construction of the quotient frame $\mathsf{Dwn}_{\mathsf{dis}}(P) \hookrightarrow \mathsf{Dwn}(P)$ is revealed to be an instance of the much more general sheafification $\mathsf{Sh}(P, J) \hookrightarrow \mathsf{PreSh}(P)$.

Let us begin with the beginning; throughout, P will stand for a meet-semilattice even though some notions still make sense for more general kinds of posets.

Definition 2. A thin presheaf[3] on P is a monotonic mapping φ: $P^{\mathsf{op}} \to 2$.

The category (which is actually just a poset, order being given pointwise) of thin presheaves on P is thus $\mathsf{Monot}(P^{\mathsf{op}}, 2)$; in the following it will be denoted as $2\text{-}\mathsf{PreSh}(P)$. Referring to the terminology of the previous section, we have that

$$2\text{-}\mathsf{PreSh}(P) \cong \mathsf{Dwn}(P).$$

This isomorphism is like the assignment of characteristic maps: for any $\varphi \in 2\text{-}\mathsf{PreSh}(P)$, $\varphi^{-1}(1) \in \mathsf{Dwn}(P)$; and given a downset D of P, define $\varphi_D : P^{\mathsf{op}} \to 2$ by putting $\varphi_D(x) = 1$ iff $x \in D$.

For every element $x \in P$ there is a representable thin presheaf (it is "represented by x")

$$\varphi_x : P^{\mathsf{op}} \to 2 : y \mapsto 1 \quad \text{iff} \quad x \le y,$$

and by the Yoneda embedding of a poset $(P, \le)$ is meant the monotonic injection

$$\mathrm{Y} : P \hookrightarrow 2\text{-}\mathsf{PreSh}(P) : x \mapsto \varphi_x. \tag{3}$$

In other words, the representable presheaves are the characteristic maps of the principal downsets of P, and the Yoneda embedding in diagram (3) is the same thing as the inclusion of the elements of the poset P into its downsets, as principal ideals, see diagram (1).

Consider next a function J_{dis} which assigns to each $x \in P$ the collection of "distributive covers of x" in P:

$J_{\mathsf{dis}}(x) = \{\{x_i\}_{i \in I} \mid \{x_i\}_{i \in I}$ is a family in P that happens to have x as its distributive join$\}$.

This is an example of a "topology" on the meet-semilattice P, in the following sense.

[3] These "thin presheaves" and the "thin sheaves," encountered further on, can be considered as examples of "enriched (pre)sheaves" (Borceux and Quinteiro, 1996).

Definition 3. A topology J on a meet-semilattice P is a function assigning to each $x \in P$ a collection $J(x)$ of families $\{x_i\}_{i \in I}$ (such families are referred to as "J-covering families of x") satisfying the following conditions:

(o) if $\{x_i\}_{i \in I} \in J(x)$ then $x_i \leq x$ for all x_i;
(i) for every $x \in P$, the singleton $\{x\}$ is an element of $J(x)$;
(ii) if $\{x_i\}_{i \in I} \in J(x)$ and $y \leq x$, then $\{x_i \wedge y\}_{i \in I} \in J(y)$;
(iii) if $\{x_i\}_{i \in I} \in J(x)$ and, for every $i \in I$, $\{x_{ik}\}_{k \in K_i} \in J(x_i)$, then $\{x_{ik}\}_{i \in I, k \in K_i} \in J(x)$.

The couple (P, J) is said to be a site.

Given a site (P, J), we are now interested in the "J-continuous" presheaves on P; they are called "sheaves."

Definition 4. A thin sheaf on a site (P, J) is a presheaf $\varphi : P^{\mathsf{op}} \to 2$ such that for every $x \in P$ and every $\{x_i\}_{i \in I} \in J(x)$, whenever $\varphi(x_i) = 1$ for all $i \in I$ then also $\varphi(x) = 1$.

The obvious poset of thin sheaves will be written as $2\text{-}\mathsf{Sh}(P, J)$, and clearly it is a subposet of $2\text{-}\mathsf{PreSh}(P)$. The topology J is said to be subcanonical if every representable presheaf is in fact a sheaf. In this case, the meet-semilattice P can be embedded in the poset $2\text{-}\mathsf{Sh}(P, J)$ by sending $x \in P$ to $\varphi_x \in 2\text{-}\mathsf{Sh}(P, J)$; this embedding is thus a corestriction of the Yoneda embedding.

The topology J_{dis} on P is easily seen to be subcanonical, and

$$2\text{-}\mathsf{Sh}(P, J_{\mathsf{dis}}) \cong \mathsf{Monot}_{\mathsf{dis}}(P^{\mathsf{op}}, 2) \cong \mathsf{Dwn}_{\mathsf{dis}}(P).$$

It can already be observed that the isomorphisms $2\text{-}\mathsf{Sh}(P) \cong \mathsf{Dwn}_{\mathsf{dis}}(P)$ and $2\text{-}\mathsf{PreSh}(P) \cong \mathsf{Dwn}(P)$ and the quotient diagram (2) exhibit the frame completion that we are interested in as corestriction of the Yoneda embedding

$$P \overset{\mathrm{Y}}{\hookrightarrow} 2\text{-}\mathsf{Sh}(P, J_{\mathsf{dis}}) \underset{\hookrightarrow}{\twoheadleftarrow} 2\text{-}\mathsf{PreSh}(P), \tag{4}$$

this corestriction $P \hookrightarrow 2\text{-}\mathsf{Sh}(P, J_{\mathsf{dis}})$ thus preserving not only all meets but also all distributive joins that happen to exist in P. We can interpret that this corestriction is obtained by "sheafifying" the thin presheaves with the aid of the topology of distributive covers on P.

The distinctive feature of the topology J_{dis} among all topologies that may exist on P, is that it is the *canonical* (i.e. finest subcanonical) topology on P; this proves the universality of the frame completion in diagram (4). We proceed with two simple lemmas, before proving this result.

Lemma 1. *Condition (ii) in the definition of "topology" is equivalent to the definition of "distributive family," in the following sense: a family $\{x_i\}_{i \in I}$ in P has a distributive join iff (a) its join exists and (b) for every $y \leq \vee_i x_i$ in P,*

$y = \vee_i(x_i \wedge y)$ *(so in particular the right-hand side of this equation is required to exist).*

Proof: Easy calculation. □

Lemma 2. *Let J' be a subcanonical topology on P. Then $\{x_i\}_{i\in I} \in J'(x)$ implies that $x = \vee_i x_i$ (and in particular the right side of this equation exists).*

Proof: Let $\{x_i\}_{i\in I} \in J'(x)$, then $x_i \leq x$ for all x_i, so surely x is an upper bound of its covering family. Suppose that x is not the smallest upper bound, then there must exist $y < x$ in P such that $x_i \leq y$ for all x_i. But the topology J' is subcanonical, so in particular the representable φ_y is a sheaf. This implies, since $\varphi_y(x_i) = 1$ for all x_i, that $\varphi_y(x) = 1$. This is in contradiction with the hypothesis that $y < x$. □

Proposition 3. *The topology of distributive covers, J_{dis}, is the finest subcanonical topology on P (and therefore it is "canonical").*

Proof: It was already observed that J_{dis} is indeed subcanonical. Suppose that J' is another subcanonical topology on P. Lemma 2 shows that the J'-covers of an element x are necessarily families $\{x_i\}_{i\in I}$ whose join (exists and) is equal to x. But Lemma 1 then says that the join of such a covering family is necessarily distributive! And therefore we must have that $J'(x) \subseteq J_{\mathsf{dis}}(x)$ for every x, meaning that the topology of distributive covers J_{dis} is finer than any such subcanonical topology J'. □

The topology of distributive covers on a meet-semilattice is thus canonical with respect to *thin* sheaves, i.e. monotonic maps $\varphi : P^{\mathsf{op}} \to 2$. But in fact this topology is canonical even with respect to sheaves into Set, i.e. functors $F : P^{\mathsf{op}} \to \mathsf{Set}$, in the appropriate topos-theoretic sense. (There exist different—but equivalent!—definitions for the notion of topology on a category. According to some references the definition we use here means that J is only a *basis* for a topology. But, as can be expected from that terminology, every basis defines a unique topology, so that no real confusion can arise.)

Theorem 1. *The topology of distributive covers is the canonical Grothendieck topology on a meet-semilattice (viewed as small category).*

Proof: First of all, our Definition 3 of (basis for a) topology on a meet-semilattice is a precise translation of the pertinent definition of (basis for a) Grothendieck topology on a small category with pullbacks, so the couple (P, J_{dis}) is really a site in the topos-theoretic sense. Next, one has to convince oneself of the fact that the representable presheaves on P, because P is a thin category, take as their values in Set either the empty set $\emptyset$ or the typical singleton $\{*\}$. That is to say, the image of a representable presheaf is contained in a category equivalent to 2. Asking for a representable presheaf, say $P(-, x) : P^{\mathrm{op}} \to \mathsf{Set}$, to be a sheaf with respect to the topology J_{dis}, is to ask for a unique amalgamation for each compatible family of elements in the sets in the image. But this notion coincides, modulo an equivalence of the image category, with our Definition 4 of thin sheaf, in the following sense: $P(-, x) : P^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf on the site (P, J_{dis}) iff $\varphi_x : P^{\mathrm{op}} \to 2$ is a thin sheaf on that site. Therefore the Lemmas 1 and 2 prove our claim. $\square$

Let us denote by $\mathsf{Sh}(P, J_{\mathsf{dis}})$, respectively $\mathsf{PreSh}(P)$, the topos of sheaves on the site (P, J_{dis}), respectively that of presheaves on P. The topology J_{dis} being canonical implies that the topos $\mathsf{Sh}(P, J_{\mathsf{dis}})$ is the smallest corestriction of the Yoneda embedding $\mathrm{Y} : P \hookrightarrow \mathsf{PreSh}(P)$ of which P can still be considered a subcategory; the corestriction itself is given by the "sheafification" of presheaves, i.e. an inclusion functor $\mathsf{Sh}(P, J_{\mathsf{dis}}) \hookrightarrow \mathsf{PreSh}(P)$ that has an essentially surjective left adjoint which itself preserves (besides all colimits also) finite limits:

$$P \overset{\mathrm{Y}}{\hookrightarrow} \mathsf{Sh}(P, J_{\mathsf{dis}}) \underset{\hookrightarrow}{\overset{\twoheadleftarrow}{}} \mathsf{PreSh}(P). \tag{5}$$

Diagram (4) is the "localic reflection" of diagram (5) in the terminology of MacLane and Moerdijk (1992).

ACKNOWLEDGMENTS

This work must be seen as complementary to a recent reformulation by J. Harding (private communication) of an older result of G. Bruns and H. Lakser, subsequently used by B. Coecke to establish a quantum logic "with an intuitionistic implication." It was after discussions with category theorist M. Jibladze that it occurred to me that the crucial ingredient of Bruns and Lakser (1970) and a fortiori (Coecke, 2001) is the canonical topology on a meet-semilattice. I have presented this paper at the Fifth Biennial Conference of the International Quantum Structures Association in Cesenatico, Italy, March 31 – April 5, 2001. I am grateful to the guest editor of this journal to have included it in this volume.

REFERENCES

Aerts, D. (1982). Description of many separated physical entities without the paradoxes encountered in quantum mechanics. *Foundations of Physics* **12**, 1131–1170.

Borceux, F. (1994). *Handbook of Categorical Algebra*, Cambridge University Press, Cambridge, UK.

Borceux, F. and Quinteiro, C. (1996). A theory of enriched sheaves. *Cahiers de Topologie et Géométrie Différentielle. Catégoriques* **37**, 145–162.

Borceux, F. and Stubbe, I. (2000). Short introduction to enriched categories. *Fundamental Theories of Physics* **111**. *Current Research in Operational Quantum Logic: Algebras, Categories, Languages*, Kluwer Academic, Dordrecht, The Netherlands, pp. 167–194.

Bruns, G. and Lakser, H. (1970). Injective hulls of semilattices. *Canadian Mathematical Bulletin* **13**, 115–118.

Coecke, B. (2001). Quantum logic in intuitionistic perspective. *Studia Logica* **70**, 411–440.

Coecke, B., Moore, D., and Stubbe, I. (2001). Quantaloids describing causation and propagation for physical entities. *Foundations of Physics Letters* **14**, 133–145.

Coecke, B. and Stubbe, I. (1999a). On a duality of quantales emerging from an operational resolution. *International Journal of Theoretical Physics* **38**, 3269–3281.

Coecke, B. and Stubbe, I. (1999b). Operational resolution and state transitions in a categorical setting. *Foundations of Physics Letters* **12**, 29–49.

Dowker, C. H. and Papert, D. (1966). Quotient frames and subspaces. *Proceedings of the London Mathematical Society* **16**, 275–296.

Johnstone, P. T. (1982). *Stone Spaces*, Cambridge University Press, Cambridge, UK.

MacLane, S. (1971). *Categories for the Working Mathematician*, Springer, Berlin.

MacLane, S. and Moerdijk, I. (1992). *Sheaves in Geometry and Logic*, Springer, Berlin.

Moore, D. (1995). Categories of representations of physical systems. *Helvetica Physica Acta* **68**, 658–678.

Piron, C. (1972). Survey of general quantum physics. *Foundations of Physics* **2**, 287–314.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, New York.

Piron, C. (1990). *Mécanique quantique.* Bases et applications, Presses Polytechniques et Universitaires Romandes, Lausanne.

Stubbe, I. (2001). *Logique quantique opérationelle et catégories*, Compte-rendu du Séminaire Itinérant de Catégories, LAMFA, Université de Picardie-Jules Verne Amiens, Vol. 18, pp. 3–14.